Math 246B Lecture 23 Notes

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1 The Montel-Caratheodory Theorem and Corollaries of Picard's Great Theorem

1.1 Proof of Schottky's theorem, continued

Last time, we were proving Schottky's theorem. Let's finish the proof.

Theorem 1.1 (Schottky). *For each $0 < \alpha < \infty$ and $0 \leq \beta < 1$, there exists a constant $M(\alpha, \beta) > 0$ such that if $f \in \mathrm{Hol}(D)$ omits the values $0, 1$ and $|f(0)| \leq \alpha$, then $|f(z)| \leq M(\alpha, \beta)$ for all $|z| \leq \beta$.*

Proof. It suffices to show this for when $\alpha \geq 2$.

Case 1: $1/2 \leq |f(0)| \leq \alpha$: We have shown that we can write $f = -\exp(i\pi \cosh(2g(z)))$ with $g \in \mathrm{Hol}(D)$, $|g(0)| \leq C(\alpha)$, and $|g'(z)| \leq C_0/(1-\beta)$ for $|z| \leq \beta < 1$, for some absolute constant C_0. Writing $g(z) = g(0) = \int_0^1 zg'(tz)\, dt$, we get

$$|g(z)| \leq C(\alpha) + \frac{C_0|z|}{1-\beta} \leq C(\alpha, \beta), \qquad |z| \leq \beta < 1.$$

We get

$$|f(z)| \leq e^{\pi e^{2|g(z)|}} \leq M(\alpha, \beta).$$

Case 2: $0 < |f(0)| < 1/2$: Apply case 1 to the function $1-f$. Then $1/2 \leq |1-f(0)| \leq 2$. So, by case 1, $|1-f(z)| \leq M(2, \beta)$ for $|z| \leq \beta < 1$. $\square$

1.2 The Montel-Caratheodory theorem

Definition 1.1. Let $\Omega \subseteq \mathbb{C}$ be open, and let $\mathcal{F} \subseteq \mathrm{Hol}(\Omega)$. We say $\mathcal{F}$ is **normal** if each sequence in $\mathcal{F}$ has a subsequence which either converges locally uniformly in $\mathrm{Hol}(\Omega)$ or tends to ∞ uniformly on each compact set.

Theorem 1.2 (Montel-Caratheodory). *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $\mathcal{F} \subseteq \mathrm{Hol}(\Omega)$ be such that for any $f \in \mathcal{F}$, $f(\Omega)$ omits the values 0,1. Then $\mathcal{F}$ is normal.*

Proof. Let (f_n) be a sequence in $\mathcal{F}$. It suffices to show that for any open disc D with $\overline{D} \subseteq \Omega$, there exists a subsequence of (f_n) which converges uniformly on D or tends to ∞ unformly on D. Let $(D_\nu)_{\nu=1}^\infty$ be such that $\overline{D}_\nu \subseteq \Omega$, $\Omega = \bigcup_{\nu=1}^\infty D_\nu$. Passing to a suitable diagonal subsequence (g_n) of (f_n), we get that for all ν, (g_n) converges uniformly on D_ν or tends to ∞ uniformly on D_ν. Let Ω_1 be the set of $z \in \Omega$ such that (g_n) converges uniformly in a neighborhood of z, and let Ω_2 be the set of $z \in \Omega$ such that (g_n) tends to ∞ uniformly in a neighborhood of z. Then Ω_1, Ω_2 are open and disjoint, and $\Omega = \Omega_1 \cup \Omega_2$, so the connectedness of Ω gives $\Omega = \Omega_1$ or $\Omega = \Omega_2$. In the first case, (g_n) converges locally uniformly in Ω, and in the second case, (g_n) tends to ∞ locally uniformly.

Let $D \subseteq \Omega$ be an open disc, and let us show that (f_n) has a subsequence which converges in $\mathrm{Hol}(D)$ or tends to ∞ locally uniformly in D. Let $D = D(z_0, R)$. We split into cases:

1. $|f_n(z_0)| \leq 1$ for infinitely many values of n: By Schottky's theorem, we get a subsequence (f_{n_j}) such that for any compact $K \subseteq D$, $|f_{n_j}(z)| \leq C_K$ for $z \in K$, $j = 1, 2, \ldots$. By Montel's theorem, we get a locally uniformly convergent subsequence.

2. $1 < |f_n(z_0)|$ for infinitely many values of n: Then apply Schottkey's theorem and then Montel's theorem to $1/f_n(z) \in \mathrm{Hol}(D)$. We get a subsequence $1/f_{n_k} \to g \in \mathrm{Hol}(D)$ locally uniformly. We have that g is either nonvanishing (then $f_{n_k} \to 1/g$ locally uniformly) or $g \equiv 0$ (then $f_{n_k} \to \infty$ locally uniformly). $\square$

1.3 Corollaries of Picard's great theorem

Recall the Casorati-Weierstrass theorem.

Theorem 1.3 (Casorati-Weierstrass)**.** *Let $a \in \mathbb{C}$, and let $f \in \mathrm{Hol}(\{0 < |z - a| < \delta\})$ have an essential singularity at a. Then the range $f(\{0 < |z - a| < \delta\})$ is dense in $\mathbb{C}$.*

Picard's great theorem is a generalization of this.

Theorem 1.4. *Let $a \in \mathbb{C}$, and let $f \in \mathrm{Hol}(\{0 < |z - a| < \delta\})$ have an essential singularity at a. There exists $w \in \mathbb{C}$ be such that the range $f(\{0 < |z - a| < r\})$ contains $\mathbb{C} \setminus \{w\}$ for all $0 < r \leq \delta$.*

Remark 1.1. The function $f(z) = e^{1/z} \neq 0$ has an essential singularity at 0.

We will prove the result next time. Here are some corollaries.

Corollary 1.1. *Let $f \in \mathrm{Hol}(\mathbb{C})$ not be a polynomial. Then for all $R > 0$, f assumes all values in $\mathbb{C}$ with at most 1 exception in $|z| > R$.*

Proof. Apply Picard's great theorem to $f(1/z)$. $\square$

Corollary 1.2. *Let f be meromorphic in $\mathbb{C}$, and suppose f is not a rational function. Then for all $R > 0$, f assumes all values in $\mathbb{C}$ with at most 2 exceptions in $|z| > R$.*

Proof. Assume that f omits 3 distinct values a, b, c in $|z| > R$. Let $g(z) = 1/(f(z) - c)$. Then g removable singularities, so it extends to an entire function. Moreover, g is not a polynomial. g omits the values $1/(a - c)$ and $1/(b - c)$ in $|z| > R$, which contradicts the previous corollary. $\square$